Sturm-Liouville

$$y'' + \lambda y = 0 \qquad a<x<b$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

BCs: $a=0, \quad b=3$

$\alpha_1 = 1, \quad \alpha_2 = 0$

$\beta_1 = 5, \quad \beta_2 = 10$

goal: find λ_n and y_n

express some function using the y_n

$$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \qquad (\lambda > 0)$$

$$y(0) = 0 \rightarrow A = 0$$

$$y = B\sin(\sqrt{\lambda}x) \qquad y' = \sqrt{\lambda}B\cos(\sqrt{\lambda}x)$$

$$5y(3) + 10y'(3) = 0$$

$$y(3) + 2y'(3) = 0$$

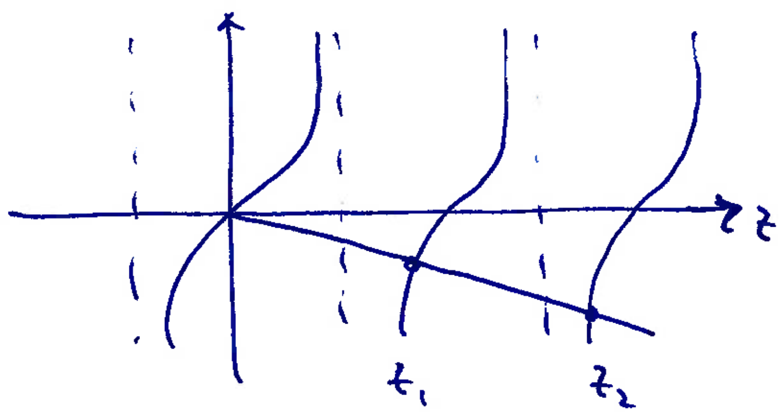
$$B\sin(3\sqrt{\lambda}) + 2\sqrt{\lambda}B\cos(3\sqrt{\lambda}) = 0 \qquad B \neq 0, \ \lambda \neq 0$$

$$\sin(3\sqrt{\lambda}) + 2\sqrt{\lambda}\cos(3\sqrt{\lambda}) = 0$$

$$\sin(3\sqrt{\lambda}) = -2\sqrt{\lambda}\cos(3\sqrt{\lambda})$$

$$\tan(3\sqrt{\lambda}) = -2\sqrt{\lambda}$$ let $z = 3\sqrt{\lambda}$

$$\tan(z) = -\frac{2}{3}z$$ intersections (positive) of $\tan(z)$ and $-\frac{2}{3}z$



$z = 3\sqrt{\lambda}$

$$\lambda_n = \frac{z_n^2}{9} \quad n = 1, 2, 3, \ldots$$

$$y_n = \sin\left(\frac{z_n}{3}x\right)$$

can λ be 0 ?

$y'' + \lambda y = 0 \quad \rightarrow \quad y = Ax + B$

$y(0) = 0 \quad \rightarrow \quad B = 0$

$y(3) + 2y'(3) = 0 \quad \rightarrow \quad 3A + 2A = 0 \quad \rightarrow \quad A = 0$ trivial solution, so $\lambda \neq 0$

eigenfunction $\quad y_n = \sin\left(\frac{\pi n}{3} x\right) = \sin(\sqrt{\lambda}\, x)$

mutually orthogonal : $\quad \int_0^3 y_n y_m \, dx = 0 \qquad n \neq m$

we can express any function using these eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} c_n y_n$$

find c_n :

multiply by y_m

$$f(x) y_m = \sum_{n=1}^{\infty} c_n y_n y_m$$

$$\int_0^3 f(x) y_m \, dx = \underbrace{\int_0^3 \sum_{n=1}^{\infty} c_n y_n y_m \, dx}_{\text{all } 0 \text{ except } n=m}$$

$$\int_0^3 f(x) y_n \, dx = \int_0^3 c_n (y_n)^2 \, dx \quad \longrightarrow \quad c_n = \frac{\int_0^3 f(x) y_n \, dx}{\int_0^3 (y_n)^2 \, dx}$$

Heat eq

$$u_t = u_{xx} \qquad 0<x<2$$

$$u(0,t)=0$$

$$u(2,t)=0$$

$$u(x,0)=f(x)$$

$$\frac{X''}{X} = + \frac{T'}{T} = -\lambda$$

$$X'' + \lambda X = 0 \qquad X(0) = X(2) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{4} \qquad X_n = \sin\left(\frac{n\pi}{2}x\right)$$

$$T' + \lambda T = 0$$

$$T' + \frac{n^2\pi^2}{4} T = 0 \qquad T = A e^{-\frac{n^2\pi^2}{4}t} \qquad T_n = e^{-\frac{n^2\pi^2}{4}t}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2/4\, t} \sin\left(\frac{n\pi}{2}x\right)$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2}x\right)$$ Sine series $$C_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx$$

Insulated / steady-state

$$u_t = u_{xx} \qquad 0<x<1$$

$$u_x(0,t) = 0$$

$$u_x(1,t) = 0$$

$$u(x,0) = f(x)$$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$X'' + \lambda X = 0 \qquad X'(0) = X'(1) = 0$$

$$X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \qquad (\lambda > 0)$$

$$X' = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$0 = B\sqrt{\lambda} \rightarrow B = 0$$

$$0 = -A\sqrt{\lambda}\sin(\sqrt{\lambda})$$

$$\sin(\sqrt{\lambda}) = 0 \rightarrow \sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n = n^2\pi^2$$

$$X_n = \cos(n\pi x)$$

if $\lambda = 0$, $\quad X'' = 0 \qquad X'(0) = X'(1) = 0$

$$X = Ax + B \qquad \Rightarrow A = 0$$

$$X' = A \qquad \lambda = 0$$

$$X = 1$$

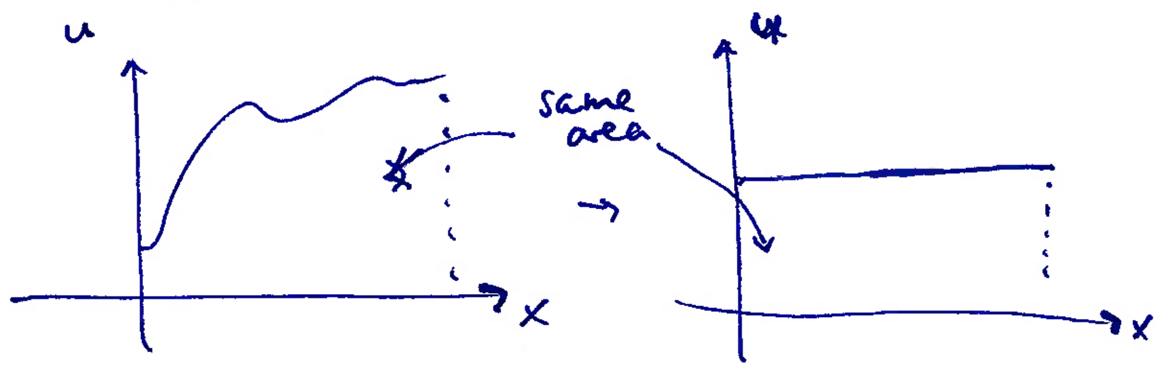
$$u(x,t) = \frac{1}{2} C_0 \cdot 1 + \sum_{n=1}^{\infty} C_n e^{-n\pi t} \cos(n\pi x)$$

$$f(x) = u(x,0) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} C_n \cos(n\pi x)$$

$$C_n = \frac{2}{1} \int_0^1 f(x) \cos(n\pi x)\, dx$$

steady-state: $t \to \infty \qquad e^{-n\pi t} \to 0$

$u \to \frac{1}{2} C_0$ average value of initial heat distribution



Laplace transform

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty u(x,t)\underbrace{e^{-st}dt}_{\text{transform away } t}$$

$$\mathcal{L}\{u\} = U(x,s)$$

$$\mathcal{L}\{u_t(x,t)\} = sU(x,s) - u(x,0)$$ (just like $\mathcal{L}\{y'\} = sY - y(0)$)

$$\mathcal{L}\{u_{tt}\} = s^2U - su(x,0) - u_t(x,0)$$

$$\mathcal{L}\{u_x\} = \frac{d}{dx}U$$

$$\mathcal{L}\{u_{xx}\} = \frac{d^2}{dx^2}U$$

example heat eq: $u_t = 3u_{xx} \quad 0 < x < \infty$

$u(x,0) = f(x) = 1$

$u(0,t) = 0$

temp is finite at right "end"

$$\mathcal{L}\{u_t\} = \mathcal{L}\{3u_{xx}\}$$

$$sU - u(x,0) = 3U'' \rightarrow \quad 3U'' - sU = -1$$

solve $\quad 3U'' - sU = 0$

$$U'' - \frac{s}{3}U = 0 \qquad U = Ae^{\sqrt{\frac{s}{3}}x} + Be^{-\sqrt{\frac{s}{3}}x}$$

Left BC: $\quad u(0,t) = 10$

$$U(x=0) = \frac{10}{s} \qquad = A + B = \frac{10}{s}$$

Right BC: bounded so $A = 0 \qquad B = \frac{10}{s}$

$$U = \frac{10}{s}e^{-\sqrt{\frac{s}{3}}x}$$ solution in s-domain

then nonhomogeneous part

then inverse transform